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A Common Fixed Point Theorem for Two Pairs of Maps Satisfying the Property (E.A)

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Abstract

In 2003, K. Jha, R.P. Pant and S.L. Singh have established in [5] a common fixed point for two pairs of compatible maps under a contractive condition of Meir-Keeler type and Lipschitz type condition. In 2008, this theorem was extended by H. Bouhadjera and A. Djoudi (see [3]) to two pairs of weakly compatible maps without using continuity. The aim of this paper is to extend the results of [5], [3] and others to the case of two pairs of occasionally weakly compatible mappings such that one of them is satisfying the property (E.A). Here, we drop the Meir-Keeler type condition and keep only the Lipschitz type condition, which, if the Lipschitz constant $k \geq 1/5$, then it is not a contractive condition of the classical type. So our approach provides some new results to the field of metric fixed point theory.

Keywords: *Common fixed points for four maps, weakly compatible maps, occasionally weakly compatible maps, noncompatible maps, property (E.A), Meir-Keeler type contractive condition, Lipschitz type condition.*

Introduction

A large part of recent metric fixed point theory is devoted to the case of four self-mappings of a metric space satisfying some contractive type conditions with some other additional assumptions.

Let (X, d) be a metric space and let A, B, S and T be four self-mappings of (X, d).

To simplify notations, for all $x, y \in X$, we set

$$m(x,y) := \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Sx,By)+d(Ax,Ty)}{2}\}$$

and

$$\sigma(x,y) := d(Sx,Ty) + d(Ax,Sx) + d(By,Ty) + d(Sx,By) + d(Ax,Ty).$$

A Meir-Keeler type (ϵ, δ)-contractive condition for the mappings A, B, S and T may be given in the form:

given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq m(x,y) < \epsilon + \delta \Longrightarrow d(Ax,By) < \epsilon \tag{1}$$

In connection to the Meir-Keeler type (ϵ, δ)-contractive condition, we consider the following two conditions:

given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X

$$\epsilon < m(x,y) < \epsilon + \delta \Longrightarrow d(Ax,By) \leq \epsilon, \tag{2}$$

and

$$d(Ax,By) < m(x,y), \quad \text{whenever} \quad m(x,y) > 0 \tag{3}$$

Jachymski [4] has shown that contractive condition (1) implies (2) but contractive condition (2) does not imply the contractive condition (1). Also, it is easy to see that the contractive condition (1) implies (3).

Condition (1) is not sufficient to ensure the existence of common fixed points of the maps A, B, S and T. Some kinds of commutativity or compatibility between the maps are always required. Also, other topological conditions on the maps or on their ranges are invoked.

Two self-mappings A and S of a metric space (X, d) are called compatible (see Jungck [7]) if,

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some t in X.

This concept was frequently used to prove existence theorems in common fixed point theory. The study on common fixed point theory for noncompatible mappings is also interesting. Work along these lines has been recently initiated by Pant [10], [11], [12].
In 2002, Aamri and Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 1. Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Remark 1. It is clear that two self-mappings of a metric space (X, d) will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$ but

$$\lim_{n\to\infty} d(STx_n, TSx_n)$$

is either non-zero or not exists.
Therefore two noncompatible self-mappings of a metric space (X, d) satisfy property (E.A).

Definition 2 ([8]). Two self mappings S and T of a metric space (X, d) are said to be weakly compatible if $Tu = Su$, for some $u \in X$, then $STu = TSu$.

It is obvious that compatibility implies weak compatibility. Examples exist to show that the converse is not true.
In 2008, Al-Thagafi and Naseer Shahzad [2] introduced the concept of occasionally weakly compatible mappings.

Definition 3. Let X be a nonempty set and T, S self-mappings on X.
A point $x \in X$ is called a coincidence point of T and S if $Tx = Sx$.
A point $w \in X$ is called a point of coincidence of T and S if there exists a coincidence point $x \in X$ of T and S such that $w = Tx = Sx$.

Definition 4. Two self-maps T and S of a nonempty set X are are called occasionally weakly compatible maps (shortly owc) [2] if there exists a point x in X which is a coincidence point for T and S at which T and S commute.

We say also that the pair (T, S) is occasionally weakly compatible.

Remark 2. Two weakly compatible mappings having coincidence points are occasionally weakly compatible. In [2], it was shown that the converse is not true.

In [5], K. Jha, R.P. Pant and S.L. Singh have established the following theorem.

Theorem 1 ([5]). Let (A, S) and (B, T) be two compatible pairs of self-mappings of a complete metric space (X, d) such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq m(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) < \epsilon, \quad \text{and}$$

(iii) $d(Ax, By) < k\sigma(x, y)$ for all $x, y \in X$, for $0 \leq k \leq \frac{1}{3}$.

If one of the mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

In [6], K. Jha has proved the following result.

Theorem 2 ([6]). Let A, B, S and T be self-mappings of a metric space (X, d) such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, y in X,

$$\epsilon < m(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) \leq \epsilon, \text{and}$$

(iii) $d(Ax, By) < k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)]$, for $0 \leq k \leq \frac{1}{3}$.
If one of AX, BX, SX and TX is a complete subspace of X and if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

In [3], Theorem 1. was generalized to the case of two pairs of weakly compatible maps by the following result.

Theorem 3 ([3]). Let (A, S) and (B, T) be two weakly compatible pairs of self-mappings of a complete metric space (X, d) such that

(a) $AX \subseteq TX$ and $BX \subseteq SX$,

(b) one of AX, BX, SX or TX is closed,

(c) given $\epsilon > 0$ there exists a $\delta > 0$ such that $\epsilon < m(x,y) < \epsilon + \delta \Longrightarrow d(Ax, By) \leq \epsilon$, and

(c') $x, y \in X$, $m(x,y) > 0 \Longrightarrow d(Ax, By) < m(x,y)$,

(d) $d(Ax, By) \leq k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)]$,

for $0 \leq k < \frac{1}{3}$.
Then A, B, S and T have a unique common fixed point.

Other related results to these theorems are published in [13] and [14].
The aim of this paper is to extend the three theorems recalled above to the case of two pairs of occasionally weakly compatible mappings such that one of them is satisfying the property (E.A) without using continuity. Here, we drop the Meir-Keeler type condition and keep only the Lipschitz type condition, which, if the Lipschitz constant $k \geq \frac{1}{5}$, then it is not a contractive condition of the classical type. So our work provides some new contributions to the field of metric fixed point theory.

Main result

To prove our main result we need the following lemma.

Lemma 1 (Jungck and Rhoades [9]). Let X be a nonempty set and let T and S two occasionally weakly compatible self-mappings of X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.

The main result of this paper reads as follows.

Theorem 4. Let (A, S) and (B, T) be two occasionally weakly compatible pairs of self-mappings of a metric space (X, d) such that

$(H1)$: $AX \subseteq TX$ and $BX \subseteq SX$,

$(H2)$: one of AX, BX, SX or TX is a closed subspace of (X, d),

$(H3)$: $d(Ax, By) \leq k\,\sigma(x,y)$, for all $x, y \in X$, where k is such that $0 \leq k < \frac{1}{3}$.

If one of the pairs $\{A, S\}$ or $\{B, T\}$ satisfies the property (E.A), then A, B, S and T have a unique common fixed point.

Proof. (I) Suppose that the pair $\{A, S\}$ satisfies the property (E.A). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \tag{4}$$

for some $z \in X$. Since $AX \subseteq TX$, then for each integer n, there exists y_n in X such that $Ax_n = Ty_n$. By using $(H3)$, we have

$$d(Ax_n, By_n)k[d(Sx_n, Ty_n) + d(Ax_n, Sx_n) + d(By_n, Ty_n) + d(Sx_n, By_n) + d(Ax_n, Ty_n)],$$

which implies

$$d(Ax_n, By_n) \leq \frac{3k}{1-2k} d(Ax_n, Sx_n). \tag{5}$$

By letting n to infinity in (4), we obtain

$$\lim_{n\to\infty} d(Ax_n, By_n) = 0. \tag{6}$$

By (4) and (6), we get

$$z = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n. \tag{7}$$

(1) Suppose that $A(X)$ is a closed subspace of (X, d). Then $z \in A(X)$. Since $AX \subseteq TX$, then there exists $u \in X$ such that $z = Tu$. By (H 3), we get

$$d(Ax_{2n}, Bu) \le k[d(Sx_{2n}, Tu) + d(Ax_{2n}, Sx_{2n}) + d(Bu, Tu) + d(Sx_{2n}, Bu) + d(Ax_{2n}, Tu)],$$

which, by letting $n \to \infty$, implies that

$$d(z, Bu) \le 2kd(z, Bu). \tag{8}$$

Since $\le k < \frac{1}{3}$, then it follows from (8) that $z = Bu$. Thus, we have $z = Tu = Bu$.

Since $B(X) \subset S(X)$, then there exists $v \in X$ such that $Bu = Sv$. Then $z = Tu = Bu = Sv$. By applying the inequality (H 3), we get

$$\begin{aligned} d(Av, Sv) &= d(Av, Bu) \\ &\le k[d(Sv, Tu) + d(Av, Sv) + d(Bu, Tu) + d(Sv, Bu) + d(Av, Tu)] \\ &= 2kd(Av, Sv), \end{aligned}$$

which implies that $Av = Sv$. Hence, we obtain

$$z = Tu = Bu = Sv = Av. \tag{9}$$

The conclusions in (9) will be obtained by similar arguments, if we suppose that $T(X)$, $B(X)$ or $S(X)$ is a closed subspace of X.

(2) By (H 3) it follows that z (given in (9)) is the unique point of coincidence for (A, S) and for (B, T). By Lemma 1. of G. Jungck and B.E. Rhoades, we conclude that z is the unique common fixed point of A, B, S and T.

(II) If we suppose that the pair $\{B, T\}$ satisfies the property (E.A), then by similar arguments we obtain the same conclusions as in the part (I).

(III) It remains to show the uniqueness of the fixed common fixed point z. Suppose that w is another common fixed point for the mappings A, B, S and T, such that $w \neq z$. Obviously we have $\sigma(w, z) = 3d(w, z) > 0$. Then, by applying the condition (H 3), we obtain

$$d(w, z) = d(Aw, Bz) \le k\sigma(w, z) = 3kd(w, z),$$

which is a contradiction. So the mappings A, B, S and T have a unique common fixed point. This completes the proof.

As a consequence, we have the following.

Corollary 1. Let (A, S) and (B, T) be two occasionally weakly compatible pairs of self-mappings of a metric space (X, d) such that

$(H1)$: $AX \subseteq TX$ and $BX \subseteq SX$,

$(H2)$: one of AX, BX, SX or TX is a closed subspace of (X, d),

$(H3)$: $d(Ax, By) \le k\,\sigma(x, y)$, for all $x, y \in X$, where k is such that $0 \le k < \frac{1}{3}$.

If one of the following two conditions is satisfied.

(i) A and S are noncompatible, or

(ii) B and T are noncompatible.

Then the mappings A, B, S and T have a unique common fixed point.

Remarks

Remarks. We observe that in both Theorem 1 and Theorem 2 the condition (iii) seems to be incorrect. The symbol " $<$ " used in this condition leads to a contradiction. Indeed, the existence of a common fixed point z in X (as asserted in both these theorems) would imply that $0 < 0$, a contradiction.

The author thinks that, it would be more convenient to replace the condition (iii) by the condition (H 3) as given in the main result of this paper to avoid contradiction. So our Theorem 4 provides a correction and some improvements to Theorem 1 and to Theorem 2.

By using a result of J. Jachymski [4], it is easy to see that the conditions (a), (c), (c') of Theorem 3 imply that the pairs (A, S) and (B, T) satisfy the property (E.A). Thus we can obtain Theorem 3. as a consequence of our Theorem 4.

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O teoremă de punct fix comun pentru două perechi de multifuncţii care satisfac proprietatea (E.A)

Rezumat

În 2003, K. Jha, R. P. Pant si S. L. Singh au demonstrat in [5] o teoremă de punct fix comun pentru două perechi de aplicaţii compatibile care satisfac o condiţie contractivă de tip Meir-Keeler si o condiţie de tip Lipschitz. In 2008, această teoremă a fost extinsă de H. Bouhadjera si A. Djoudi (vezi [3]) la două perechi de aplicaţii slab compatibile fără a folosi continuitatea. Scopul acestei lucrări este extinderea rezultatelor din [5], [3] si alte lucrări la cazul a două perechi de aplicaţii ocazional slab compatibile, dintre care una satisface condiţia (E.A). Eliminăm condiţia de tip Meir-Keeler si pastrăm numai condiţia de tip Lipschitz, care pentru constante Lipschitz $k \geq 1/5$ *nu mai este o condiţie contractivă de tip clasic. Abordarea noastră permite obţinerea unor rezultate noi in teoria punctelor fixe în spaţii metrice.*